**Least Action Principle**

Now let’s study Maxwell’s equations from a least action principle. This is beneficial in a few respects. One is that in order to quantize the EM field (the subject of quantum electrodynamics) we must be able to write the Lagrangian for the EM field. Another advantage is that doing this will introduce us to the stress-energy tensor of the EM field, which is relevant to general relativity as the contribution of the EM field to space-time curvature.

**Action for Fields (with source term j)**

So, without deriving S per se′, here is the action (in our fake Gaussian units still):



where these guys are 4-vectors. Note that this is somewhat similar to the particle action. **|F|**2 gives the squares of the fields (i.e., the energy in the fields) = FαβFαβ (implicit summation over indices). The other term, **A**·**j**, is there because it gives term gives the coupling of the fields to the particles. When we minimize the action, then we’ll recover ME. Let’s illustrate how this happens. I’ll do this in general for now. Let ℒ be the integrand of S. We’ll note that ℒ depends generically on the space-time potential components Aα and its derivatives ∂βAα­­, implicitly through F and explicitly in the source term. So then functional minimization would require (see CM folder for more on this):



There is implicit summation over the α, β indices. Continuing,



where we’ve integrated by parts in the last term to switch the derivative. Continuing,



And so finally, to put it in a form similar to the typical EL equations (remember implicit summation over β):



We can do this more explicitly, from the beginning, filling all the A’s into F. We would have:



(the factor of two is because when taking the derivative of the second term in the product we can raise the second terms indices and lower the first ones because if we do both, it won’t change the result. And in the last term we’ve integrated by parts to put the derivative on the A’s. So continuing,



which says that:



which is Maxwell’s equations just like in last file. So this action is the required one. For future reference, let’s work out the Lagrangian in more detail – just the free part.



Let’s split this term into pieces:



OK, so adding them altogether we get:



So then finally,



And so we can write the Lagrangian as:



So,



And purely in terms of the space-time potential, we’d have:



So up till now we are treating **j** as a source term that would have to be supplied.

**Action for Particles and their Fields Together**

We can generalize our action by treating both particles and fields together as mutually dependent quantities. This is also the action required to describe mutually interacting charged particles. It requires a simple change. We just add the kinetic energy of the particles, and make cosmetic changes to the source term to incorporate the position/velocity of the particles. We say,



(could generalize to different charges by e → qi) Filling this into L = ∫d3r ℒ, and adding the electron kinetic energy term, we get:



We should recognize the first two terms from Classical Mechanics. We can get the force equation by minimizing w/r to **r**i. Recall from Classical Mechanics we perform (i being particle, and α the coordinate component α = x, y, z):



Now for convenience going to take the i subscript off of ri, and note that A(ri,ti) is really A(ri(t),t) (same with φ technically but doesn’t matter, been leaving those arguments off ‘cause they’re messy), and so we have to use chain rule when taking derivative. So,



Now remember in Gaussian units E = -∇φ - ∂A/∂t, so we can say:



And we hope the last part has to do with the magnetic field,



Well let’s just show that is what we want (Einstein summation),



So it checks out, and we have:



Of course **E** and **B** are unspecified, but to get them, we’d repeat what we did above and minimize the action w/r to φ and **A**. Note we do differentiate the current terms w/r to **A** again, like we did before. We’d have for instance,



And we’d get Maxwell’s equations out of it.

**Constructing Hamiltonian in Coulomb Gauge**

Let’s construct the Hamiltonian. This is:



In our case **Q**(x,t) = (φ, A), and **q** = **r**. So working this out, using the Coulomb gauge (in Gaussian units) for convenience (I assume this would also work out the same way in the Lorentz gauge):



Putting whole thing in terms of the potentials, we have:



Then,



where **1** is the unit tensor – basically Kronecker delta. So filling that into H,



Now use that ∇2φ = -4πρ equation, along with IBP, to see that:



So can say,



Now thanks to IBP and ∇·**A** = 0 → ∂(∇·**A**)/∂t = 0 → ∇·(∂**A**/∂t) = 0 we can add **·**𝛁φ terms to the integrand,



So finally,



Well, more finally, putting v in terms of p and A,



So H clearly consists of the kinetic energy of charges, and their associated potential energy, stored in the fields.

**Constructing Hamiltonian in Lorentz Gauge**

Let’s construct the Hamiltonian in the Lorentz Gauge. This is:



In our case **Q**(x,t) = (φ, A), and **q** = **r**. So working this out, using the Coulomb Lorentz gauge (in Gaussian units):



Putting whole thing in terms of the potentials, we have, as before:



Then, getting the momentum conjugates, as before:



where **1** is the unit tensor – basically Kronecker delta. Then filling that into H, we ultimately get:



Now we diverge from our work in the Coulomb gauge. We use ∇2φ - ∂2φ/∂t2 = -4πρ equation, along with IBP, to see that:



So can say,



Now thanks to IBP and ∇·**A** = -∂φ/∂t → ∂(∇·**A**)/∂t = -∂2φ/∂t2 → ∇·(∂**A**/∂t) = - ∂2φ/∂t2, we can say,



So finally,



and then,



**Action for particles and their fields…in *external field* too**

What if have particles and their fields evolving against a backdrop of an external field? Then Lagrangian would be:



right? Pretty sure that this would give us the correct equations of motion. And then the Hamiltonian would go to:



Since φext and Aext are fixed they are not independent variables. So filling that into H,



Repeating all that stuff from before, we see this will come to:



or in other words,



and **E**, **B** are the fields created solely by the particles which are doing the moving, i.e., don’t comprise the external fields as well.

**Particles and their Fields Action → Particles Action**

If we solve the field’s equations of motion in terms of the particle and current densities, and plug this back in, we’ll have the action (or Hamiltonian) for just the particles themselves (maybe see what we did in Classical Mechanics file vis a vis the gravitational field action). First thing we’ll do is go back to Coulomb gauge guy and sort of partially undo what we did, and rewrite E in terms of the potentials:



And use this again,



To get:



Now the Coulomb gauge condition, implies the solution:



as we can verify by revisiting all that Green’s function stuff in the beginning of the file. Filling in our density this comes to:



So filling that into H we have:



and so,



Might check out the Quantum Field Theory file for what we can further do with these, once we quantize the vector field **A** (basically gives the free-photon part of H). At least in the steady state (and usually in the non-QFT scenario) we can just get away with neglecting that term. Can also check out the Electrodynamics Folder/Free EB energy thoughts to see how we can write the ∫B2 term as an integral over currents. I think this move would more formally and perspicuously follow from redoing everything in the Lorentz gauge. So could write:



But we usually ignore the ∫B2 term as being insignificant next to the power of the ∫E2 term. And so we’d just have the first two terms. If we wanted to include an external field, then from our work above, we could surmise we’d have:



**Appendix: Converting Back to MKS units**

So let’s consider what some of these guys would look like in MKS units. So remember that we need to add back in factors of (4πε­0)p and (μ0/4π)q where p and q are powers to be determined so that the different terms in the expression are dimensionally consistent with each other, and also come out to the dimensions the overall expression is supposed to have. Typically, p and q are equal, and so this reduces to some power of c (speed of light). So noting that ℒ has units of energy density, H has units of energy, we should have:

